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Distance Integral Complete r-Partite Graphs

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Abstract. Let $D(G) = (d_{ij})_{n\times n}$ denote the distance matrix of a connected graph G with order n, where d_{ij} is equal to the distance between vertices v_i and v_j in G. A graph is called distance integral if all eigenvalues of its distance matrix are integers. In this paper, we investigate distance integral complete r-partite graphs $K_{p_1,p_2,\ldots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\ldots,a_s\cdot p_s}$ and give a sufficient and necessary condition for $K_{a_1\cdot p_1,a_2\cdot p_2,\ldots,a_s\cdot p_s}$ to be distance integral, from which we construct infinitely many new classes of distance integral graphs with $s = 1, 2, 3, 4$. Finally, we propose two basic open problems for further study.

1. Introduction

Let G be a simple connected undirected graph with n vertices. The vertex set of G is denoted by $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $d_i = d(v_i)$ be the degree of the vertex v_i in G. The adjacency matrix of G, $A(G) = (a_{ij})$ is an $n \times n$ matrix, where $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. The eigenvalues of $A(G)$, labeled as $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, are said to be eigenvalues of G and form the adjacency spectrum of G. A graph is called integral if all its eigenvalues are integers. The signless Laplacian matrix of G is defined as $Q(G) = Deg(G) + A(G)$, where $Deg(G) = diag(d_1, d_2, \ldots, d_n)$ is the diagonal matrix of the vertex degrees in G. The eigenvalues of $Q(G)$ are said to be the signless Laplacian eigenvalues or Q-eigenvalues of G. A graph G is called Q-integral if all its Q-eigenvalues are integers. The notion of integral graphs was first introduced by Harary and Schwenk in 1974 [14]. The study on integral graphs and Q-integral graphs has drawn many scholars' attentions. Results about them are found in [5, 7, 8, 13–16, 22, 26, 32] and [10, 23, 29, 34], respectively.

The distance between the vertices v_i and v_j is the length of a shortest path between them, and is denoted by d_{ij}. The distance matrix of G, denoted by $D(G)$, is the $n \times n$ matrix whose (i, j)-entry is equal to d_{ij} for $i, j = 1, 2, \ldots, n$ (see [4]). Note that $d_{ii} = 0$, $i = 1, 2, \ldots, n$. The distance characteristic polynomial (or D-polynomial) of G is $D_G(x) = |xI_n - D(G)|$, where I_n is the $n \times n$ identity matrix. The eigenvalues of $D(G)$ are said to be the distance eigenvalues or D-eigenvalues of G. Since $D(G)$ is a real symmetric matrix, the D-eigenvalues are real and can be labeled as $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$. The distance spectral radius of G is the largest D-eigenvalue μ_1 and denoted by $\mu(G)$. Assume that $\mu_1 > \mu_2 > \ldots > \mu_t$ are t distinct D-eigenvalues of G with the corresponding multiplicities $k_1, k_2, \ldots, k_t$. We denote by $Spec(G) = \begin{pmatrix} \mu_t & \mu_{t-1} & \ldots & \mu_2 & \mu_1 \\ k_t & k_{t-1} & \ldots & k_2 & k_1 \end{pmatrix}$ the

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Distance spectrum or the D-spectrum of G. Similarly to integral graphs, a graph is called distance integral if all its D-eigenvalues are integers. Many results about distance spectral radius and the D-eigenvalues of graphs can be found in [1, 3, 9, 17, 19, 20, 27, 28, 31, 35, 36].

The energy of G was originally defined by Gutman in 1978 as the sum of the absolute values of the eigenvalues of $A(G)$ [11]. It is used in chemistry to approximate the total π-electron energy of molecules. Some results about graph energy can be found in [11, 12, 21, 24, 25, 28] and the book [33]. Based on the research of graph energy, the concept of distance energy of D-energy of a graph G defined as the sum of the absolute values of the eigenvalues of $D(G)$ was recently introduced by Indulal et al. in [18]. Several invariants of this type (as well as a few others) were studied by Consonni and Todeschini in [6] for possible use in QSPR modelling. Their study showed, among other things, that the distance energy is a useful molecular descriptor. Some results about D-energy can be found in [17, 18, 28, 30, 33, 36].

Our motivation for the research of distance integral graphs came from the work above. A *complete r-partite* ($r \geq 2$) *graph* $K_{p_1,p_2,\cdots,p_r}$ is a graph with a set $V = V_1 \cup V_2 \cup \cdots \cup V_r$ of $p_1 + p_2 + \cdots + p_r (= n)$ vertices, where V_i's are nonempty disjoint sets, $|V_i| = p_i$, such that two vertices in V are adjacent if and only if they belong to different V_i's. Assume that the number of distinct integers of $p_1, p_2, \cdots, p_r$ is s. Without loss of generality, assume that the first s ones are the distinct integers such that $p_1 < p_2 < \ldots < p_s$. Suppose that a_i is the multiplicity of p_i for each $i = 1, 2, \ldots, s$. The complete r-partite graph $K_{p_1,p_2,\cdots,p_r} = K_{p_1,\ldots,p_1,\ldots,p_s,\ldots,p_s}$ on n vertices is also denoted by $K_{a_1 \cdot p_1, a_2 \cdot p_2, \ldots, a_s \cdot p_s}$, where $r = \sum_{i=1}^{s} a_i$ and $n = \sum_{i=1}^{s} a_i p_i$. In this paper, we investigate distance integral complete r-partite graphs $K_{p_1,p_2,\ldots,p_r} = K_{a_1 \cdot p_1, a_2 \cdot p_2, \ldots, a_s \cdot p_s}$. We give a sufficient and necessary condition for the graph $K_{a_1 \cdot p_1, a_2 \cdot p_2, \ldots, a_s \cdot p_s}$ to be distance integral, from which we construct infinitely many new classes of such distance integral graphs with $s = 1, 2, 3, 4$. Finally, we propose two basic open problems for further study.

2. A Sufficient and Necessary Condition for Complete *r*-Partite Graphs to be Distance Integral

In this section, we shall give a sufficient and necessary condition for complete r-partite graphs to be distance integral. Similar results for integrality of complete r-partite graphs were given in [32] and for Q-integrality of complete r-partite graphs were given in [34].

The following Theorem 2.1 has already been obtained by Lin et al. in [19] and by Stevanović et al. in [30], respectively.

Theorem 2.1. *(See Theorem 4.1 of [19] or [30]) Let G be a complete r-partite graph $K_{p_1,p_2,\ldots,p_r}$ on n vertices. Then the D-polynomial of G is*

$$D_G(x) = \prod_{i=1}^{r} (x+2)^{(p_i-1)} \prod_{i=1}^{r} (x - p_i + 2)\left(1 - \sum_{i=1}^{r} \frac{p_i}{x - p_i + 2}\right). \tag{1}$$

Corollary 2.2. *Let G be a complete r-partite graph $K_{p_1,p_2,\ldots,p_r} = K_{a_1 \cdot p_1,\ldots,a_s \cdot p_s}$ on n vertices. Then the D-polynomial of G is*

$$D_G(x) = \prod_{i=1}^{s} (x+2)^{a_i(p_i-1)} \prod_{i=1}^{s} (x - p_i + 2)^{a_i}\left(1 - \sum_{i=1}^{s} \frac{a_i p_i}{x - p_i + 2}\right). \tag{2}$$

Proof. We can easily obtain the result from Theorem 2.1. □

From Corollary 2.2, we can obtain the following result.

Corollary 2.3. *For the complete r-partite graph $K_{p_1,p_2,\ldots,p_r} = K_{a_1 \cdot p_1, a_2 \cdot p_2, \ldots, a_s \cdot p_s}$ of order n, we have*

(1) If $s = 1$, then $K_{a_1 \cdot p_1} = K_{p_1,\ldots,p_1}$ is distance integral, and its D-spectrum is

$$Spec(K_{a_1 \cdot p_1}) = \begin{pmatrix} -2 & p_1 - 2 & n + p_1 - 2 \\ n - a_1 & a_1 - 1 & 1 \end{pmatrix}. \tag{3}$$

(2) If $s = 2, a_1 = a_2 = 1$, then K_{p_1,p_2} is distance integral if and only if $(p_1^2 + p_2^2 - p_1 p_2)$ is a perfect square.

Following result can also be obtained by Corollary 2.2.

Theorem 2.4. *The complete r-partite graph $K_{p_1,p_2,\dots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\dots,a_s\cdot p_s}$ on n vertices is distance integral if and only if*

$$\prod_{i=1}^{s}(x-p_i+2)-\sum_{j=1}^{s}a_jp_j\prod_{i=1,i\neq j}^{s}(x-p_i+2)=0 \tag{4}$$

has only integral roots.

We can get more information by discussing Eq.(4) of Theorem 2.4. First, we divide both sides of Eq.(4) by $\prod_{i=1}^{s}(x-p_i+2)$, and obtain the following equation.

$$\sum_{i=1}^{s}\frac{a_ip_i}{x-p_i+2}=1. \tag{5}$$

Let $F(x)=1-\sum_{i=1}^{s}\frac{a_ip_i}{x-p_i+2}$. Obviously, $x=(p_i-2)$'s are not roots of Eq.(4) for $1\le i\le s$. Hence, all solutions of Eq.(4) are the same as those of Eq.(5). Now we consider the roots of $F(x)$ over the set of real numbers. Note that $F(x)$ is discontinuous at each point $x=p_i-2$. We obtain that $\lim\limits_{x\to(p_i-2)^-}F(x)=+\infty$, $\lim\limits_{x\to(p_i-2)^+}F(x)=-\infty$, $\lim\limits_{x\to-\infty}F(x)=\lim\limits_{x\to+\infty}F(x)=1, F'(x)=\sum_{i=1}^{s}\frac{a_ip_i}{(x-p_i+2)^2}$, for $1\le i\le s$. We deduce that $F(x)$ is strictly monotone increasing on each of the continuous interval over the set of real numbers. By the Bolzano's Theorem or the Weierstrass Intermediate Value Theorem of Analysis, we get that $F(x)$ has s distinct real roots. If $-\infty<\mu_1<\mu_2<\cdots<\mu_{s-1}<\mu_s<+\infty$ are the roots of $F(x)$, then

$$-2<p_1-2<\mu_1<p_2-2<\mu_2<\cdots<p_{s-1}-2<\mu_{s-1}<p_s-2<\mu_s<+\infty \tag{6}$$

holds.

From the above discussion, we have the following result.

Theorem 2.5. *The complete r-partite graph $K_{p_1,p_2,\dots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\dots,a_s\cdot p_s}$ on n vertices is distance integral if and only if all the solutions of Eq.*(5) *are non-negative integers. Moreover, the graph $K_{p_1,p_2,\dots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\dots,a_s\cdot p_s}$ is distance integral if and only if there exist integers $\mu_1,\mu_2,\dots,\mu_s$ satisfying* (6) *such that the following linear equation system in $a_1,a_2,\dots,a_s$*

$$\begin{cases}\frac{a_1p_1}{\mu_1-p_1+2}+\frac{a_2p_2}{\mu_1-p_2+2}+\cdots+\frac{a_sp_s}{\mu_1-p_s+2}=1\\ \cdots\cdots\cdots\qquad\cdots\cdots\cdots\\ \frac{a_1p_1}{\mu_s-p_1+2}+\frac{a_2p_2}{\mu_s-p_2+2}+\cdots+\frac{a_sp_s}{\mu_s-p_s+2}=1\end{cases} \tag{7}$$

has positive integral solutions $(a_1,a_2,\dots,a_s)$.

Theorem 2.6. *If the complete r-partite graph $K_{p_1,p_2,\dots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\dots,a_s\cdot p_s}$ on n vertices is distance integral then there exist integers $\mu_i(i=1,2,\dots,s)$ such that $-2<p_1-2<\mu_1<p_2-2<\mu_2<\cdots<p_{s-1}-2<\mu_{s-1}<p_s-2<\mu_s<+\infty$ and the numbers $a_1,a_2,\dots a_s$ defined by*

$$a_k=\frac{\prod_{i=1}^{s}(\mu_i-p_k+2)}{p_k\prod_{i=1,i\neq k}^{s}(p_i-p_k)},k=1,2,\dots,s, \tag{8}$$

are positive integers.

Conversely, suppose that there exist integers μ_i $(i=1,2,\dots,s)$ such that $-2<p_1-2<\mu_1<p_2-2<\mu_2<\cdots<p_{s-1}-2<\mu_{s-1}<p_s-2<\mu_s<+\infty$ and that the numbers $a_k=\frac{\prod_{i=1}^{s}(\mu_i-p_k+2)}{p_k\prod_{i=1,i\neq k}^{s}(p_i-p_k)}(k=1,2,\dots,s)$ are positive integers. Then the complete r-partite graph $K_{p_1,p_2,\dots,p_r} = K_{a_1\cdot p_1,\dots,a_s\cdot p_s}$ is distance integral.

Proof. From Cauchy's result on determinants in [2], we know that

$$\begin{vmatrix} \frac{1}{\alpha_1+\beta_1} & \frac{1}{\alpha_1+\beta_2} & \cdots & \frac{1}{\alpha_1+\beta_s} \\ \frac{1}{\alpha_2+\beta_1} & \frac{1}{\alpha_2+\beta_2} & \cdots & \frac{1}{\alpha_2+\beta_s} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\alpha_s+\beta_1} & \frac{1}{\alpha_s+\beta_2} & \cdots & \frac{1}{\alpha_s+\beta_s} \end{vmatrix} = \frac{\prod_{1\le i<j\le s}(\alpha_j-\alpha_i)(\beta_j-\beta_i)}{\prod_{1\le i,j\le s}(\alpha_i+\beta_j)}. \tag{9}$$

The determinant of the coefficient matrix D of the linear equation system (7) is the following:

$$|D| = \begin{vmatrix} \frac{p_1}{\mu_1-p_1+2} & \frac{p_2}{\mu_1-p_2+2} & \cdots & \frac{p_s}{\mu_1-p_s+2} \\ \frac{p_1}{\mu_2-p_1+2} & \frac{p_2}{\mu_2-p_2+2} & \cdots & \frac{p_s}{\mu_2-p_s+2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{p_1}{\mu_s-p_1+2} & \frac{p_2}{\mu_s-p_2+2} & \cdots & \frac{p_s}{\mu_s-p_s+2} \end{vmatrix} = \prod_{i=1}^{s} p_i \begin{vmatrix} \frac{1}{\mu_1-p_1+2} & \frac{1}{\mu_1-p_2+2} & \cdots & \frac{1}{\mu_1-p_s+2} \\ \frac{1}{\mu_2-p_1+2} & \frac{1}{\mu_2-p_2+2} & \cdots & \frac{1}{\mu_2-p_s+2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\mu_s-p_1+2} & \frac{1}{\mu_s-p_2+2} & \cdots & \frac{1}{\mu_s-p_s+2} \end{vmatrix}$$

$$= \frac{\prod_{i=1}^{s} p_i \prod_{1\le i<j\le s}(\mu_j-\mu_i)(p_i-p_j)}{\prod_{1\le i,j\le s}(\mu_i-p_j+2)} \neq 0.$$

Moreover, for $k = 1, 2, \ldots, s$,

$$|D_k| = \begin{vmatrix} \frac{p_1}{\mu_1-p_1+2} & \frac{p_2}{\mu_1-p_2+2} & \cdots & \frac{p_{k-1}}{\mu_1-p_{k-1}+2} & 1 & \frac{p_{k+1}}{\mu_1-p_{k+1}+2} & \cdots & \frac{p_{s-1}}{\mu_1-p_{s-1}+2} & \frac{p_s}{\mu_1-p_s+2} \\ \frac{p_1}{\mu_2-p_1+2} & \frac{p_2}{\mu_2-p_2+2} & \cdots & \frac{p_{k-1}}{\mu_2-p_{k-1}+2} & 1 & \frac{p_{k+1}}{\mu_2-p_{k+1}+2} & \cdots & \frac{p_{s-1}}{\mu_2-p_{s-1}+2} & \frac{p_s}{\mu_2-p_s+2} \\ \vdots & \vdots & \ddots & \vdots & & & & & \\ \frac{p_1}{\mu_s-p_1+2} & \frac{p_2}{\mu_s-p_2+2} & \cdots & \frac{p_{k-1}}{\mu_s-p_{k-1}+2} & 1 & \frac{p_{k+1}}{\mu_s-p_{k+1}+2} & \cdots & \frac{p_{s-1}}{\mu_s-p_{s-1}+2} & \frac{p_s}{\mu_s-p_s+2} \end{vmatrix}$$

$$= -lim_{p_k\to+\infty}|D|$$

$$= \frac{\prod_{i=1,i\neq k}^{s} p_i \prod_{1\le i<j\le s,i,j\neq k}(\mu_j-\mu_i)(p_i-p_j)\prod_{i=1,i\neq k}(\mu_k-\mu_i)}{\prod_{1\le i,j\le s,j\neq k}(\mu_i-p_j+2)}.$$

By using the well-known Cramer's Rule to solve the linear equation system (7) in $a_1, a_2, \ldots, a_s$, we get that

$$a_k = \frac{|D_k|}{|D|} = \frac{\prod_{i=1}^{s}(\mu_i-p_k+2)}{p_k\prod_{i=1,i\neq k}^{s}(p_i-p_k)}, \quad (k = 1, 2, \ldots, s). \tag{10}$$

If the graph $K_{p_1,p_2,\ldots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\ldots,a_s\cdot p_s}$ is distance integral, because μ_i and a_i $(i = 1, 2, \ldots, s)$ are integers, $-2 < p_1-2 < \mu_1 < p_2-2 < \mu_2 < \cdots < p_{s-1}-2 < \mu_{s-1} < p_s-2 < \mu_s < +\infty$ and $p_i \ge 1$ for $i = 1, 2, \ldots, s$, we can deduce that $a_k > 0 (k = 1, 2, \ldots, s)$.

On the other hand, from Theorem 2.4, we obtain

$$\prod_{i=1}^{s}(x-\mu_i) = \prod_{i=1}^{s}(x-p_i+2) - \sum_{j=1}^{s} a_jp_j \prod_{i=1,i\neq j}^{s}(x-p_i+2).$$

Because $\mu_i (i = 1, 2, \ldots, s)$ are integers, from Corollary 2.2, the sufficient condition of the theorem can be easily proved. □

Corollary 2.7. *If the complete r-partite graph $K_{p_1,p_2,\ldots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\ldots,a_s\cdot p_s}$ on n vertices is distance integral with non-negative integral eigenvalues $\mu_i (i = 1, 2, \ldots, s)$ are those of Theorem 2.6, then we get the following results:*

(1) $\sum_{i=1}^{s}\mu_i = \sum_{i=1}^{s}(p_i-2)+n$*, where* $n = \sum_{i=1}^{s} a_ip_i$.

(2) $\prod_{i=1}^{s}\mu_i = \prod_{i=1}^{s}(p_i-2)(1+\sum_{i=1}^{s}\frac{a_ip_i}{p_i-2})$.

(3) $Spec(K_{a_1\cdot p_1,a_2\cdot p_2,\ldots,a_s\cdot p_s}) = \begin{pmatrix} -2 & p_1-2 & \mu_1 & p_2-2 & \ldots & \mu_{s-1} & p_s-2 & \mu_s \\ n-\sum_{i=1}^{s} a_i & a_1-1 & 1 & a_2-1 & \ldots & 1 & a_s-1 & 1 \end{pmatrix}$

Proof. From Corollary 2.2, we can get that

$$\begin{aligned} D_G(x) &= \prod_{i=1}^{s}(x+2)^{a_i(p_i-1)}\prod_{i=1}^{s}(x-p_i+2)^{a_i-1}[\prod_{i=1}^{s}(x-p_i+2)-\sum_{j=1}^{s}a_jp_j\prod_{i=1,i\neq j}^{s}(x-p_i+2)] \\ &= \prod_{i=1}^{s}(x+2)^{a_i(p_i-1)}\prod_{i=1}^{s}(x-p_i+2)^{a_i-1}\prod_{i=1}^{s}(x-\mu_i). \end{aligned}$$

By using the relationship between roots and coefficients of polynomials, we obtain the results in (1) − (3). □

In order to study the relationship between the distance integral complete r-partite graph $K_{p_1,p_2,\ldots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\ldots,a_s\cdot p_s}$ and vectors $\vec{a} = (a_1, a_2, \ldots, a_s), \vec{p} = (p_1, p_2, \ldots, p_s) \in \mathbb{Z}^s$, we have the following lemma.

Lemma 2.8. *Define*

$$\psi_{\vec{a},\vec{p}}(x) = \sum_{i=1}^{s}\frac{a_ip_i}{x-p_i+2}, \phi_{\vec{a},\vec{p}}(x) = \prod_{i=1}^{s}(x-p_i+2)(1-\psi_{\vec{a},\vec{p}}(x)),$$

where $n = \sum_{i=1}^{s} a_ip_i$, vectors $\vec{a} = (a_1, a_2, \ldots, a_s), \vec{p} = (p_1, p_2, \ldots, p_s) \in \mathbb{Z}^s$. Let q be a nonzero integer.Then μ is an integral root of $\phi_{\vec{a},q\vec{p}}(x)$ if and only if $[(\mu+2)/q]-2$ is an integral root of $\phi_{\vec{a},\vec{p}}(x)$.

Proof. It is obvious that α is a root of $\phi_{\vec{a},\vec{p}}(x)$ if and only if $q(\alpha+2)-2$ is a root of $\phi_{\vec{a},q\vec{p}}(x)$, therefore if all the roots of $\phi_{\vec{a},\vec{p}}(x)$ are integers, then the roots of $\phi_{\vec{a},q\vec{p}}(x)$ are integers as well.

Assume now that all roots of $\phi_{\vec{a},q\vec{p}}(x)$ are integral and let α be one of them, then $[(\alpha+2)/q]-2$ is a rational root of $\phi_{\vec{a},\vec{p}}(x)$. Since $\phi_{\vec{a},\vec{p}}(x)$ is a monic polynomial with integral coefficients, its rational roots should be integers. Therefore $[(\alpha+2)/q]-2 \in \mathbb{Z}$. □

Corollary 2.9. *For any positive integer q, the complete r-partite graph $K_{p_1q,\ p_2q,\ldots,p_rq} = K_{a_1\cdot p_1q,a_2\cdot p_2q,\ldots,a_s\cdot p_sq}$ is distance integral if and only if the complete r-partite graph $K_{p_1,p_2,\ldots,p_r=K_{a_1\cdot p_1,a_2\cdot p_2,\ldots,a_s\cdot p_s}}$ is distance integral.*

Remark 2.10. *Let $GCD(p_1, p_2, \ldots, p_s)$ denote the greatest common divisor of the numbers $p_1, p_2, \ldots, p_s$. We say that a vector $(p_1, p_2, \ldots, p_s)$ is primitive if $GCD(p_1, p_2, \ldots, p_s) = 1$. Corollary 2.9 shows that it is reasonable to study Eq.*(5) *only for primitive vectors $(p_1, p_2, \ldots, p_s)$.*

3. Distance Integral Complete *r*-Partite Graphs

In this section, we shall construct infinitely many new classes of distance integral complete r-partite graphs $K_{p_1,p_2,\ldots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\ldots,a_s\cdot p_s}$ with $s = 2, 3, 4$.

The main idea for constructing such distance integral graphs is as follows:

(i) We properly choose positive integers $p_1, p_2, \ldots, p_s$.

(ii) We try to find integers $\mu_i (i = 1, 2, \ldots, s)$ satisfying (6) such that there are positive integral solutions $(a_1, a_2, \ldots, a_s)$ for the linear equation system (7) (or such that all $a_k's$ of (8) are positive integers).

(iii) We can obtain integers $a_1, a_2, \ldots, a_s$ such that all the solutions of Eq. (5) are integers. Thus, we have constructed many new classes of distance integral graphs $K_{a_1\cdot p_1,a_2\cdot p_2,\ldots,a_s\cdot p_s}$.

Theorem 3.1. *For $s = 2$, let $p_1 < p_2$. Then $K_{a_1\cdot p_1,a_2\cdot p_2}$ of order n is distance integral if and only if one of the following two conditions holds:*

(i) When $GCD(p_1, p_2) = 1$, let $\mu_1 = p_1 + q - 2, 1 \le q < p_2 - p_1$, where q is a positive integer. Then, a_1 and a_2 must be the positive integral solutions for the Diophantine equation

$$qp_2a_2 + p_1(p_1 - p_2 + q)a_1 = q(p_1 - p_2 + q). \tag{11}$$

(ii) When $GCD(p_1, p_2) = d \geq 2$, let $p_1 = p'_1 d, p_2 = p'_2 d, GCD(p'_1, p'_2) = 1, \mu_1 = p_1 + q - 2, q = q'd, 1 \leq q' < p'_2 - p'_1$, where p'_1, p'_2, q' and d are positive integers. Then, a_1 and a_2 must be positive integral solutions for the Diophantine equation

$$q'p'_2 a_2 + p'_1(p'_1 - p'_2 + q')a_1 = q'(p'_1 - p'_2 + q'). \tag{12}$$

Proof. Since $p_1 < p_2$, from Theorem 2.6, we know $K_{a_1 \cdot p_1, a_2 \cdot p_2}$ is distance integral if and only if there exist integers μ_1, μ_2 and positive integers p_1, p_2 such that $-2 < p_1 - 2 < \mu_1 < p_2 - 2 < \mu_2$ and

$$a_1 = \frac{(\mu_1 - p_1 + 2)(\mu_2 - p_1 + 2)}{p_1(p_2 - p_1)}, a_2 = \frac{(\mu_1 - p_2 + 2)(\mu_2 - p_2 + 2)}{p_2(p_1 - p_2)}$$

are positive integers.

Hence, we choose $\mu_1 = p_1 + q - 2, 1 \leq q < p_2 - p_1$, where q is a positive integer, and we obtain

$$a_1 = \frac{q(\mu_2 - p_1 + 2)}{p_1(p_2 - p_1)}, a_2 = \frac{(p_1 - p_2 + q)(\mu_2 - p_2 + 2)}{p_2(p_1 - p_2)}$$

Then, we get Eq.(11). From elementary number theory, we know there are solutions for Eq.(11) if and only if $d_1 | q(p_1 - p_2 + q)$, where $d_1 = GCD(qp_2, p_1(p_1 - p_2 + q))$.

Now, we discuss two cases.

Case 1. When $GCD(p_1, p_2) = 1$, we have $d_1 | q(p_1 - p_2 + q)$. Moreover, there are solutions for Eq.(11). From elementary number theory and the condition $GCD(p_1, p_2) = 1$, we know that there are infinitely many integral solutions for Eq.(11). Therefore, there are infinitely many positive integral solutions (a_1, a_2) for Eq.(11).

Case 2. When $GCD(p_1, p_2) = d \geq 2$, let $p_1 = p'_1 d, p_2 = p'_2 d, GCD(p'_1, p'_2) = 1$, where p'_1, p'_2 and d are positive integers. We have $d_1 = GCD(qp_2, p_1(p_1 - p_2 + q)) = GCD(qp'_2 d, p'_1 d(p'_1 d - p'_2 d + q))$. If $d_1 | q(p_1 - p_2 + q) = q(p'_1 d - p'_2 d + q)$, then $d | q$. Thus, let $q = q'd, 1 \leq q' < (p'_2 - p'_1)$, where q' is a positive integer. We can reduce (11) and (12). Hence, from elementary number theory and the condition $GCD(p'_1, p'_2) = 1$, we know that there are infinitely many integral solutions for Eq.(12). Therefore, there are infinitely many positive integral solutions (a_1, a_2) for Eq.(12). □

Theorem 3.2. *Let a complete r-partite graph $K_{p_1, p_2, \ldots, p_r} = K_{a_1 \cdot p_1, a_2 \cdot p_2, \ldots, a_s \cdot p_s}$ be distance integral with eigenvalues μ_i. Let $\mu_i (\geq 0)$ and $p_i (> 0) (i = 1, 2, \ldots, s)$ be integers such that $-2 < p_1 - 2 < \mu_1 < p_2 - 2 < \mu_2 < \cdots < p_{s-1} - 2 < \mu_{s-1} < p_s - 2 < \mu_s < +\infty$ and*

$$a_k = \frac{\prod_{i=1}^{s}(\mu_i - p_k + 2)}{p_k \prod_{i=1, i \neq k}^{s}(p_i - p_k)}, k = 1, 2, \ldots, s \tag{13}$$

are positive integers, then for

$$b_k = \frac{\prod_{i=1}^{s-1}(\mu_i - p_k + 2)(\mu_s - p_k + 2 + rt)}{p_k \prod_{i=1, i \neq k}^{s}(p_i - p_k)}, k = 1, 2, \ldots, s, \tag{14}$$

$$r = LCM(r_1, r_2, \ldots, r_s), r_k = \frac{p_k \prod_{i=1, i \neq k}^{s}(p_i - p_k)}{d_k}, k = 1, 2, \ldots, s, \tag{15}$$

$$d_k = GCD(\prod_{i=1}^{s-1}(\mu_i - p_k + 2), p_k \prod_{i=1, i \neq k}^{s}(p_i - p_k)), k = 1, 2, \ldots, s, \tag{16}$$

the complete m-partite graph $K_{p_1, p_2, \ldots, p_m} = K_{b_1 \cdot p_1, b_2 \cdot p_2, \ldots, b_s \cdot p_s}$ is distance integral for every nonnegative integer t with eigenvalues $\mu_1, \mu_2, \ldots, \mu_{s-1}, \mu'_s = \mu_s + rt$. (Similar results for integral complete multipartite graphs were given in [16])

Proof. From (14) for every $k = 1, 2, \dots, s$ after simplification we get $b_k = a_k + \frac{rt\prod_{i=1}^{s-1}(\mu_i - p_k + 2)}{p_k\prod_{i=1,i\neq k}^{s}(p_i - p_k)}$. Since $r = LCM(r_1, r_2, \dots, r_s), r_k = \frac{p_k\prod_{i=1,i\neq k}^{s}(p_i - p_k)}{d_k}$, b_k is an integer for every $k = 1, 2, \dots, s$. Let us denote $\mu_s' = \mu_s + rt$. As $\mu_s \leq \mu_s' < +\infty$, by Theorem 2.6 the graph $K_{p_1,p_2,\dots,p_m} = K_{b_1\cdot p_1, b_2\cdot p_2,\dots,b_s\cdot p_s}$ is distance integral. □

Theorem 3.3. *For $s = 3$, integers $p_i(> 0)$, $a_i(> 0)$ and $\mu_i (i = 1, 2, 3)$ are given in Table 1. p_i, a_i and $\mu_i (i = 1, 2, 3)$ are those of Theorem 2.6, where $(p_1, p_2, p_3) = 1$. Then for any positive integer q the graph $K_{a_1\cdot p_1 q, a_2\cdot p_2 q, a_3\cdot p_3 q}$ on n vertices is distance integral.*

Table 1: Distance integral complete r-partite graphs $K_{a_1\cdot p_1, a_2\cdot p_2, a_3\cdot p_3}$.

p_1	p_2	p_3	a_1	a_2	a_3	μ_1	μ_2	μ_3	p_1	p_2	p_3	a_1	a_2	a_3	μ_1	μ_2	μ_3
1	6	14	6	1	3	0	5	64	2	4	9	6	2	2	1	4	42
2	5	9	6	2	4	1	4	63	2	5	15	8	5	2	1	8	78
2	6	15	10	7	4	1	8	130	2	7	11	5	2	4	1	6	75
2	9	12	4	2	3	1	8	70	2	9	13	9	7	5	1	9	154
2	9	17	7	1	1	3	10	49	2	10	17	7	4	6	1	10	168
2	12	21	6	3	6	1	12	190	3	5	12	4	1	4	2	4	73
3	5	12	8	4	2	2	7	73	3	7	12	8	6	6	2	7	145
3	7	15	5	1	2	3	7	61	3	8	12	2	1	2	2	7	46
3	8	12	7	1	1	4	8	46	3	8	14	8	8	5	2	9	166
3	9	19	7	9	3	2	13	169	3	10	15	3	1	1	3	10	43
3	11	21	7	1	1	5	13	64	3	12	20	4	4	4	2	13	154
4	6	15	10	6	2	3	10	112	4	7	12	9	1	5	4	6	110
4	9	18	8	3	4	4	10	142	4	10	15	2	2	2	3	10	68
4	11	16	4	1	3	4	10	86	4	12	15	10	7	3	4	12	178
5	8	12	7	6	5	4	8	150	5	9	12	4	3	5	4	8	115
5	9	18	5	7	2	4	13	133	5	10	18	2	3	1	4	13	68
5	11	20	7	2	2	6	13	108	5	12	17	3	2	1	5	13	66
5	12	18	2	3	2	4	13	94	5	12	20	10	3	4	6	13	178
5	13	17	9	6	5	5	13	219	5	14	20	6	1	4	6	13	138
5	15	24	3	6	3	4	18	193	6	9	14	2	2	1	5	10	52
6	11	13	8	10	9	5	10	284	6	13	22	5	10	5	5	16	284
6	14	21	2	4	2	5	16	124	6	15	22	9	7	7	6	16	328
7	10	20	8	9	6	6	13	278	7	11	17	9	4	4	7	12	185
7	11	21	8	3	6	7	12	229	7	15	18	9	5	9	7	14	313
7	15	25	7	4	2	8	19	173	7	16	24	2	5	2	6	19	158
8	11	13	9	2	3	8	10	141	8	11	16	7	1	6	8	10	174
8	11	18	2	2	2	7	12	86	8	13	20	4	3	1	8	16	102
8	15	24	5	1	2	10	16	118	8	18	27	2	6	2	7	22	196
9	12	20	3	4	1	8	16	106	9	15	20	1	2	1	8	16	73
9	15	25	3	7	3	8	19	223	9	16	20	3	6	5	8	16	238
9	16	20	8	2	6	10	15	238	9	16	20	9	4	4	10	16	238
9	16	20	10	6	2	10	17	238	9	19	26	7	6	2	10	22	245
10	13	22	2	2	3	9	14	128	10	15	24	4	3	2	10	18	148
10	17	26	6	4	1	11	22	168	/	/	/	/	/	/	/	/	/

Proof. From Theorem 2.6, we know that the complete multipartite graph $K_{a_1\cdot p_1, a_2\cdot p_2,\dots,a_s\cdot p_s}$ on n vertices is distance integral if and only if there exist integers μ_i and positive integers p_i $(i = 1, 2, \dots, s)$ such that (6) holds and $a_k = \frac{\prod_{i=1}^{s}(\mu_i - p_k + 2)}{p_k\prod_{i=1,i\neq k}^{s}(p_i - p_k)}$ $(k = 1, 2, \dots, s)$ are positive integers.

By Corollary 2.7(1), we know $\sum\limits_{i=1}^{3}\mu_i = \sum\limits_{i=1}^{3}(p_i - 2) + n$, where $n = \sum_{i=1}^{3} a_i p_i$. It deduces that $\mu_3 = -\mu_1 - \mu_2 + \sum\limits_{i=1}^{3}(p_i - 2) + n$.

From Corollary 2.9, we need only consider the case $(p_1, p_2, \dots, p_s) = 1$. Hence, when $s = 3$, it is sufficient to find only all positive integers p_i, a_i $(i = 1, 2, 3)$, μ_1, μ_2 and μ_3 for the following equations:

$$a_1 = \frac{(\mu_1 - p_1 + 2)(\mu_2 - p_1 + 2)(-\mu_1 - \mu_2 + p_2 + p_3 + n - 4)}{p_1(p_2 - p_1)(p_3 - p_1)}, \tag{17}$$

$$a_2 = \frac{(\mu_1 - p_2 + 2)(\mu_2 - p_2 + 2)(-\mu_1 - \mu_2 + p_1 + p_3 + n - 4)}{p_2(p_1 - p_2)(p_3 - p_2)}, \tag{18}$$

$$a_3 = \frac{(\mu_1 - p_3 + 2)(\mu_2 - p_3 + 2)(-\mu_1 - \mu_2 + p_1 + p_2 + n - 4)}{p_3(p_2 - p_3)(p_1 - p_3)}. \tag{19}$$

By using a computer search, we have found 67 integral solutions satisfying $GCD(p_1, p_2, p_3) = 1$ for (17), (18) and (19). They are listed in Table 1, where $1 \le p_1 \le 10$, $p_1 + 1 \le p_2 \le p_1 + 10$, $p_2 + 1 \le p_3 \le p_2 + 10$, $1 \le a_1 \le 10$, $1 \le a_2 \le 10$, $1 \le a_3 \le 10$, $p_1 - 2 < \mu_1 < p_2 - 2$, $p_2 - 2 < \mu_2 < p_3 - 2$, and $n = \sum_{i=1}^{3} a_i p_i$. By Corollary 2.9, it follows that these graphs $K_{a_1 \cdot p_1 q, a_2 \cdot p_2 q, a_3 \cdot p_3 q}$ are distance integral for any positive integer q. □

Theorem 3.4. *For $s = 3$, let $p_i(> 0)$, $a_i(> 0)$ and $\mu_i(> 0)$ $(i = 1, 2, 3)$ be those of Theorem 2.6. Then for any positive integer q, the graphs $K_{a_1 \cdot p_1 q, a_2 \cdot p_2 q, \ldots, a_3 \cdot p_3 q}$ with n vertices are distance integral if $p_1 = 2$, $p_2 = 4$, $p_3 = 9$, $a_1 = 20t + 6$, $a_2 = 7t + 2$, $a_3 = 8t + 2$, $\mu_1 = 1$, $\mu_2 = 4$, $\mu_3 = 140t + 42$, and $n = \sum_{i=1}^{3} a_i p_i = 140t + 38$, where t is a nonnegative integer.*

Proof. The proof directly follows from Theorem 3.2 and Corollary 2.9. □

Remark 3.5. *Similarly to Theorem 3.4 it is possible to find conditions for parameters $\mu_3, a_i (i = 1, 2, 3)$ which depend on t for each graph in Table 1. In this way we get new classes of distance integral graphs.*

Theorem 3.6. *For $s = 4$, integers $p_i(> 0)$, $a_i(> 0)$ and $\mu_i(i = 1, 2, 3)$ are given in Table 2. p_i, a_i and $\mu_i(i = 1, 2, 3, 4)$ are those of Theorem 2.6, where $GCD(p_1, p_2, p_3, p_4) = 1$. Then for any positive integer q the graph $K_{a_1 \cdot p_1 q, a_2 \cdot p_2 q, a_3 \cdot p_3 q, a_4 \cdot p_4 q}$ on n vertices is distance integral.*

Table 2: Distance integral complete r-partite graphs $K_{a_1 \cdot p_1, a_2 \cdot p_2, a_3 \cdot p_3, a_4 \cdot p_4}$.

p_1	p_2	p_3	p_4	a_1	a_2	a_3	a_4	μ_1	μ_2	μ_3	μ_4
1	6	13	22	95	5	10	1	2	6	19	284
1	6	22	40	183	16	8	2	2	12	34	548
1	8	21	26	171	5	2	8	4	11	20	474
1	9	16	36	115	15	1	8	2	13	19	574
1	9	25	36	32	7	3	1	1	14	31	223
1	11	16	26	56	2	2	2	4	11	19	174
1	11	17	27	93	1	1	2	7	12	19	185
1	11	19	24	196	3	2	2	7	14	20	321
1	11	22	36	256	6	7	2	6	14	31	559
1	16	31	56	108	10	1	3	4	26	39	494
1	16	31	56	135	14	2	1	4	26	49	494
2	5	17	21	33	13	21	6	1	5	18	627
2	8	17	35	114	23	8	2	3	12	30	627
3	8	15	21	107	7	6	8	5	10	16	643
3	15	19	33	47	3	1	2	9	16	25	283
4	12	19	24	68	8	2	11	7	14	18	682
4	16	34	49	24	14	3	2	5	26	42	542
5	8	19	21	27	6	6	6	5	10	18	435
5	9	17	35	28	4	1	6	6	13	19	423
7	11	16	20	13	2	2	1	8	12	17	174
7	22	31	52	30	10	4	2	11	26	45	680
9	18	34	42	9	2	1	3	12	22	34	304
13	22	29	38	25	5	5	1	17	24	35	636
15	24	35	42	15	4	1	1	19	31	38	418

Proof. From Theorem 2.6, we know that the complete multipartite graph $K_{a_1\cdot p_1,a_2\cdot p_2,\ldots,a_s\cdot p_s}$ on n vertices is distance integral if and only if there exist integers μ_i and positive integers p_i $(i = 1, 2, \ldots, s)$ such that (6) holds and $a_k = \frac{\prod_{i=1}^{s}(\mu_i - p_k + 2)}{p_k \prod_{i=1,i\neq k}^{s}(p_i - p_k)}$ $(k = 1, 2, \ldots, s)$ are positive integers.

By Corollary 2.7(1), we know $\sum\limits_{i=1}^{4} \mu_i = \sum\limits_{i=1}^{4} (p_i - 2) + n$, $n = \sum_{i=1}^{4} a_i p_i$. It deduces that

$$\mu_4 = -\mu_1 - \mu_2 - \mu_3 + \sum_{i=1}^{4} (p_i - 2) + n.$$

From Corollary 2.9, we need only consider the case $(p_1, p_2, \ldots, p_s) = 1$. Hence, when $s = 4$, it is sufficient to find only all positive integers p_i, a_i $(i = 1, 2, 3, 4)$, μ_1, μ_2, μ_3 and μ_4 for the following equations:

$$a_1 = \frac{[\prod\limits_{i=1}^{3}(\mu_i - p_1 + 2)](\sum\limits_{i=1}^{4} (p_i - 2) + n - \mu_1 - \mu_2 - \mu_3 - p_1 + 2)}{p_1(p_2 - p_1)(p_3 - p_1)(p_4 - p_1)}, \tag{20}$$

$$a_2 = \frac{[\prod\limits_{i=1}^{3}(\mu_i - p_2 + 2)](\sum\limits_{i=1}^{4} (p_i - 2) + n - \mu_1 - \mu_2 - \mu_3 - p_2 + 2)}{p_2(p_1 - p_2)(p_3 - p_2)(p_4 - p_2)}, \tag{21}$$

$$a_3 = \frac{[\prod\limits_{i=1}^{3}(\mu_i - p_3 + 2)](\sum\limits_{i=1}^{4} (p_i - 2) + n - \mu_1 - \mu_2 - \mu_3 - p_3 + 2)}{p_3(p_1 - p_3)(p_2 - p_3)(p_4 - p_3)}, \tag{22}$$

$$a_4 = \frac{[\prod\limits_{i=1}^{3}(\mu_i - p_4 + 2)](\sum\limits_{i=1}^{4} (p_i - 2) + n - \mu_1 - \mu_2 - \mu_3 - p_4 + 2)}{p_4(p_1 - p_4)(p_2 - p_4)(p_3 - p_4)}. \tag{23}$$

By using a computer search, we have found 24 integral solutions satisfying $GCD(p_1, p_2, p_3, p_4) = 1$ for (20), (21), (22) and (23). They are listed in Table 2, where $1 \le p_1 \le 15$, $p_1+1 \le p_2 \le p_1+15$, $p_2+1 \le p_3 \le p_2+20$, $p_3+1 \le p_4 \le p_3+25$, $\mu_4 < 700$, $p_1-2 < \mu_1 < p_2-2$, $p_2-2 < \mu_2 < p_3-2$, $p_3-2 < \mu_3 < p_4-2$, and $n = \sum_{i=1}^{4} a_i p_i$. By Corollary 2.9, it follows that these graphs $K_{a_1\cdot p_1 q, a_2\cdot p_2 q, a_3\cdot p_3 q, a_4\cdot p_4 q}$ are distance integral for any positive integer q. □

According to Theorem 3.2, we can obtain the following theorem.

Theorem 3.7. *For $s = 4$, let $p_i(> 0)$, $a_i(> 0)$ and $\mu_i(> 0)$ $(i = 1, 2, 3, 4)$ be those of Theorem 2.6. Then for any positive integer q, the complete multipartite graphs $K_{a_1\cdot p_1 q, a_2\cdot p_2 q, a_3\cdot p_3 q,\ldots,a_4\cdot p_4 q}$ with n vertices are distance integral if $p_1 = 1$, $p_2 = 6$, $p_3 = 13$, $p_4 = 22$, $\mu_1 = 2$, $\mu_2 = 6$, $\mu_3 = 19$ and $\mu_4 = 24024t + 284$, $a_1 = 8008t + 95$, $a_2 = 429t + 5$, $a_3 = 880t + 10$ and $a_4 = 91t + 1$, $n = \sum_{i=1}^{4} a_i p_i = 24024t + 277$, where t is a nonnegative integer.*

Proof. The proof directly follows from Theorem 3.2 and Corollary 2.9. □

Remark 3.8. *Similarly to Theorem 3.7 it is possible to find conditions for parameters $\mu_4, a_i (i = 1, 2, 3, 4)$ which depend on t for each graph in Table 2. In this way we get new classes of distance integral graphs.*

4. Further Discussion

For complete r-partite graphs $K_{p_1,p_2,\ldots,p_r}$=$K_{a_1\cdot p_1,a_2\cdot p_2,\ldots,a_s\cdot p_s}$, when $s = 1, 2, 3, 4$, we have obtained such distance integral graphs in this paper. However, when $s \ge 5$, we have not found such distance integral graphs. We tried to get some general results. Thus we raise the following questions:

Question 4.1. *Are there any distance integral complete r-partite graphs* $K_{p_1,p_2,\ldots,p_r} = K_{a_1\cdot p_1,\ldots,a_s\cdot p_s}$ *when* $s \geq 5$*?*

For complete r-partite graphs $K_{p_1,p_2,\ldots,p_r}$=$K_{a_1\cdot p_1,a_2\cdot p_2,\ldots,a_s\cdot p_s}$, when $s = 1, 2$, and a_1=$a_2 = \ldots = a_s = 1$, some results about such distance integral graphs have been obtained in this paper. However, when $s \geq 3$, $a_1 = a_2 = \ldots = a_s = 1$, we have not found such distance integral graphs. Hence, we raise the following question:

Question 4.2. *Are there any distance integral complete r-partite graphs* $K_{p_1,p_2,\ldots,p_r} = K_{a_1\cdot p_1,a_2\cdot p_2,\ \ldots,\ a_s\cdot p_s}$ *with* a_1= $a_2 = \ldots = a_s = 1$ *when* $s \geq 3$*?*

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